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Combined Systems in Quantum Probability

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A framework for quantum probability is developed and combinations of systems are studied within this framework. In particular, we consider the horizontal sum, the direct sum, and the Cartesian product of quantum probability systems. The relations between these combinations and the concepts of interference and independence of measurements are derived. We also consider the amplitude superselection structure of these combinations.

1. INTRODUCTION

The quantum probability framework that we shall use is based on ideas of Feynman (1948; Feynman and Hibbs, 1965; Schulman, 1981). According to Feynman, at any given time, a physical system S is in precisely one state (or configuration or alternative) ω and each state has an amplitude for occurring. If X is a measurement on S, then executing X results in a unique outcome depending on the state ω of S. The amplitude of an outcome x of a measurement X is the "sum" of the amplitudes of all states that result in x upon executing X. The probability of an outcome of X is the modulus squared of its amplitude. We have developed these ideas in previous works (Gudder, 1988*a*,*b*, 1989, and to appear) and refer the reader to these for a fuller discussion.

The three main ingredients of a quantum probability theory for S are the set of states Ω, a set of measurements $\mathscr{A}$ which we call a catalog, and the set of amplitude densities $\mathscr{D}(\mathscr{A})$ for $\mathscr{A}$. This paper is primarily concerned with various ways of combining catalogs and amplitude densities. In particular, if $\mathscr{A}_1$ and $\mathscr{A}_2$ are catalogs, we define their horizontal sum $\mathscr{A}_1+\mathscr{A}_2$, direct sum $\mathscr{A}_1\oplus\mathscr{A}_2$, and Cartesian product $\mathscr{A}_1\mathscr{A}_2$. Corresponding to these combinations, there are natural ways of combining their amplitude densities.

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For $f_1 \in \mathscr{D}(\mathscr{A}_1)$ and $f_2 \in \mathscr{D}(\mathscr{A}_2)$ we define the combinations $f_1 \circ f_2$ on $\mathscr{A}_1 + \mathscr{A}_2$, $f_1 \oplus f_2$ on $\mathscr{A}_1 \oplus \mathscr{A}_2$, and $f_1 f_2$ on $\mathscr{A}_1 \mathscr{A}_2$. Moreover, we introduce the concepts of interference and independence of measurements relative to an amplitude density and study the manifestations of these concepts for the various combinations. We also consider superpositions of amplitude densities. These result in (unnormalized) amplitudes, and we call maximal superposition sets of amplitudes, sectors. The sector structures for combined systems are developed. In order to give simple motivations of the theory without measure-theoretic technicalities, finite quantum probability models are developed in Section 2. Section 3 studies horizontal sums within the finite model. In Section 4, we present the full mathematical theory as motivated by Section 2. Finally, Sections 5 and 6 develop the theory of direct sums and Cartesian products.

2. FINITE MODELS

Since a full formulation of quantum probability theory requires certain measure-theoretic technicalities, it is instructive to first consider a simple finite model. Let $\Omega = \{\omega_1, \ldots, \omega_n\}$ be the set of possible states for a physical system S. Let $f: \Omega \to \mathbb{C}$ be an amplitude function, where $f(\omega_j)$ gives the amplitude that the state ω_j occurs, $j = 1, \ldots, n$. If X is a measurement for S, we denote by $X(\omega) = x$ the outcome resulting when X is executed and S is in the state $\omega \in \Omega$. Thus, we can consider X as a function $X: \Omega \to R(X)$, where the range $R(X)$ is the set of X-outcomes. The amplitude of the outcome x upon executing X is defined by

$$f_X(x) = \sum \{f(\omega): X(\omega) = x\} = \sum_{\omega \in X^{-1}(x)} f(\omega) \tag{2.1}$$

We call $f_X: R(X) \to \mathbb{C}$ the *(X, f)-wave function.* Notice that (2.1) is the prescription for obtaining the amplitude of x given in the introduction. The *probability* of x upon executing X is defined as $P_{X,f}(x) = |f_X(x)|^2$. Finally, if $B \subseteq R(X)$ is a set of X-outcomes (an *X-event*), the probability of B is given by

$$P_{X,f}(B) = \sum_{x \in B} P_{X,f}(x) = \sum_{x \in B} |f_X(x)|^2 \tag{2.2}$$

In order for $P_{X,f}$ to be a probability distribution, we must assume that f satisfies the normalization condition

$$\sum_{x \in R(X)} |f_X(x)|^2 = 1 \tag{2.3}$$

If (2.3) holds, we call f an *amplitude density* for X. We then call $P_{X,f}$ the *f-distribution* of X.

Although this formalism describes quantum mechanical situations, the reader should notice that we have not begun with the Hilbert space of states and have not defined an observable as a self-adjoint operator. These traditional quantum mechanical constructs are derived from our more primitive axioms. In fact, due to the normalization condition (2.3), the wave function f_X is a unit vector in the Hilbert space $H_X = l^2(R(X))$. Traditional quantum mechanics begins with the wave function $f_X: R(X) \to \mathbb{C}$ and misses the underlying "reality" given by the space Ω. Moreover, traditional quantum mechanics describes the measurement X by the operator $\tilde{X}$ on H_X given by $(\tilde{X}f)(x) = xf(x)$. This replaces the function X by the more complicated and less precise operator $\tilde{X}$.

So far we have only discussed a single measurement and events corresponding to that measurement. In quantum mechanics it is important to consider several measurements simultaneously and their corresponding events. For this reason, we introduce a nonempty collection of measurements $\mathscr{A}$. We denote by $\mathscr{D}(\mathscr{A})$ the set of functions that are simultaneous amplitude densities for all the measurements in $\mathscr{A}$. Let $X \in \mathscr{A}$, let $A \subseteq \Omega$, and suppose we want to obtain information about A by employing the measurement X. In general, A may have no relationship to X. In fact, A is frequently of the form $A = Y^{-1}(B)$, where $Y \in \mathscr{A}$ is another measurement. In this case, we seek information about Y using a different measurement X. We define the amplitude that A occurs and X results in x by

$$f_X(A)(x) = \sum \{f(\omega): \omega \in A \cap X^{-1}(x)\} \tag{2.4}$$

In particular, notice that for every $B \subseteq R(X)$ we have

$$f_X[X^{-1}(B)] = \chi_B f_X \tag{2.5}$$

where χ_B is the characteristic function for B, so (2.4) is a reasonable definition. Moreover, when $A = \Omega$ we have $f_X(\Omega) = f_X$. It is then natural to define the (X, f)*-pseudoprobability* of A as

$$P_{X,f}(A) = \sum_{x \in R(X)} |f_X(A)(x)|^2 \tag{2.6}$$

As a special case, applying (2.5), this gives for $B \subseteq R(X)$

$$P_{X,f}(B) \equiv P_{X,f}[X^{-1}(B)] = \sum_{x \in R(X)} \chi_B(x)|f_X(x)|^2 = \sum_{x \in B} |f_X(x)|^2$$

which is consistent with (2.2).

Although $P_{X,f}(A)$ is always nonnegative, we call it a pseudoprobability, since it may be larger than 1 and is not generally additive in A. For these reasons, $P_{X,f}(A)$ may not be interpretable as a probability. However, in many situations it does have the properties of a true probability distribution. For example, we have seen this to be the case for $A = X^{-1}(B)$, $B \subseteq R(X)$.

What is the operational meaning of $P_{X,f}(A)$ as given in (2.6)? That is, how does one calculate $P_{X,f}(A)$ using laboratory data? First, A must be a physical event; that is, one which is preparable in the laboratory. Prepare the event A a large number N times, and each time A is prepared, execute the measurement X. Let $R(X)=\{x_1,\ldots,x_m\}$ and suppose x_j results n_j times, $j=1,\ldots,m$. Thus, $\sum n_j=N$. If A is preparable, then its complement A' should also be be preparable. Follow the same procedure for A' and suppose x_j now results n_j' times, $j=1,\ldots,m$. Then the number $n_j/(n_j+n_j')$ gives the probability $P_{X,f}(A|x_j)$ that A occurs given that x_j occurs when X is executed. Now the probability $P_{X,f}(x_j)$ is easily calculated. Simply perform X a large number N times and divide the number of occurrences r_j of x_j by N. These numbers are then used to calculate

$$P_{X,f}(A)=\sum_{j=1}^{m} P_{X,f}(x_j)P_{X,f}(A|x_j)=\frac{1}{N}\sum_{j=1}^{m}\frac{r_jn_j}{n_j+n_j'}$$

Now it is an axiom of quantum probability that this expression and the one in (2.6) coincide.

Now let $B\subseteq R(X)$, $A\subseteq\Omega$ and $P_{X,f}(A)\neq 0$, and define the conditional probability

$$P_{X,f}(B|A)=\frac{P_{X,f}[X^{-1}(B)\cap A]}{P_{X,f}(A)} \tag{2.7}$$

It is easy to show that

$$f_X[X^{-1}(B)\cap A]=\chi_B f_X(A) \tag{2.8}$$

Applying (2.8), (2.7) becomes

$$P_{X,f}(B|A)=\frac{1}{P_{X,f}(A)}\sum_{x\in B}|f_X(A)(x)|^2 \tag{2.9}$$

Notice that $B\mapsto P_{X,f}(B|A)$ is a true probability measure on the set of X-events. For X, $Y\in\mathscr{A}$ and $f\in\mathscr{D}(\mathscr{A})$, we say that X is *f-independent* of Y if

$$\sum_{x\in B}|f_X[Y^{-1}(A)](x)|^2=P_{X,f}(B)P_{X,f}[Y^{-1}(A)] \tag{2.10}$$

for every $B\subseteq R(X)$, $A\subseteq R(Y)$. Applying (2.9) and (2.10), we conclude that

$$P_{X,f}(B|Y^{-1}(A))=P_{X,f}(B)$$

for every $B\subseteq R(X)$, $A\subseteq R(Y)$ with $P_{X,f}[Y^{-1}(A)]\neq 0$. We say that X *does not interfere with* Y *relative to* f if

$$P_{X,f}[Y^{-1}(A)]=P_{Y,f}(A) \tag{2.11}$$

for every $A\subseteq R(Y)$.

Our definition of independence is analogous to the classical definition and does not need further justification. However, our definition of noninterference is fairly recent, so we shall discuss it in some detail. [A different justification is given in Gudder (1989 and 1990).] We first give a geometric interpretation. Suppose for simplicity that

$$\Omega = \{\omega_{ij} : i, j = 1, \ldots, n\}$$

Let X, Y be measurements defined by $X(\omega_{ij}) = x_i$ and $Y(\omega_{ij}) = y_j$, $i, j = 1, \ldots, n$. Suppose X does not interfere with Y relative to f and let $A = \{y_1, y_2\} \subseteq R(Y)$. Since

$$Y^{-1}(A) = \{\omega_{i1}, \omega_{i2} : i = 1, \ldots, n\}$$

it follows from (2.11) that

$$\sum_{i=1}^{n} |f(\omega_{i1}) + f(\omega_{i2})|^2 = \left|\sum_{i=1}^{n} f(\omega_{i1})\right|^2 + \left|\sum_{i=1}^{n} f(\omega_{i2})\right|^2 \tag{2.12}$$

Visualizing Ω as a matrix, on the left side of (2.12) we first sum along a row, take the modulus squared, and then sum these, while on the right side, we first sum along a column, take the modulus squared, and then sum these. Hence, the orders of summation and of taking the modulus squared are different. A similar equality must hold for every $A \subseteq R(Y)$. We thus see that two measurements do not interfere only under very special circumstances.

Section 4 generalizes the present one by allowing measures on the sets $X^{-1}(x)$, $x \in R(X)$. For the present finite case, this entails the introduction of nonnegative weights w_i corresponding to the elements $\omega_i \in X^{-1}(x)$. The wave function at x then becomes

$$f_X(x) = \sum \{w_i f(\omega_i) : \omega_i \in X^{-1}(x)\}$$

We now analyze the concept of noninterference in traditional quantum mechanics. For simplicity, suppose our system is described by a finite-dimensional Hilbert space H. Let Ω be the unit sphere $S(H)$ of H. This is the usual quantum mechanical state space. Of course, in this case Ω is no longer finite. However, we shall overcome this difficulty by placing weights so that only a finite number of states have nonzero weight. For $\psi \in S(H)$ define the amplitude $f = f_\psi : \Omega \to \mathbb{C}$ by $f(\phi) = \langle \psi, \phi \rangle$. Let $\{\phi_1, \ldots, \phi_n\}$ be an orthonormal basis in $S(H)$ and let $\lambda_1, \ldots, \lambda_n, \lambda_\infty$ be distinct real numbers. Define $X : \Omega \to \mathbb{R}$ by

$$X(\phi) = \begin{cases} \lambda_i & \text{if} \quad \phi = \phi_i, \quad i = 1, \ldots, n \\ \lambda_\infty & \text{otherwise} \end{cases}$$

Place the standard counting measure on $X^{-1}(\lambda_i)$, $i=1,\ldots,n$, and place the zero measure on $X^{-1}(\lambda_\infty)$. Then X is a measurement corresponding to an observable that has the value λ_i in the state ϕ_i, $i=1,\ldots,n$. Notice that f is an amplitude density for X, since $f_X(\lambda_i)=\langle\psi,\phi_i\rangle$, $i=1,\ldots,n$, and $f_X(\lambda_\infty)=0$, so

$$\sum_{\lambda\in R(X)}|f_X(\lambda)|^2=\sum_{j=1}^{n}|\langle\psi,\phi_i\rangle|^2=\|\psi\|^2=1$$

Similarly, let $\{\phi'_1,\ldots,\phi'_n\}$ be an orthonormal basis in $S(H)$ and let $\lambda'_1,\ldots,\lambda'_n,\lambda'_\infty$ be distinct real numbers. Define $X':\Omega\to\mathbb{R}$ in an analogous way. Suppose X does not interfere with X'. Then

$$P_{X,f}[(X')^{-1}(A)]=P_{X',f}(A)$$

for every $A\subseteq R(X')$. Letting $A=\{\lambda'_i\}$, we have

$$\sum_{j=1}^{n}|f(\phi_j)|^2\delta_{\phi_j\phi'_i}=|f(\phi'_i)|^2$$

Hence, if $\psi\not\perp\phi'_i$, then $\phi_j=\phi'_i$ for some $j\in\{1,\ldots,n\}$. Therefore, if $\psi\not\perp\phi'_i$ for every $i=1,\ldots,n$, then $\{\phi_1,\ldots,\phi_n\}=\{\phi'_1,\ldots,\phi'_n\}$. Since the ϕ_i correspond to eigenvectors, this condition characterizes commuting self-adjoint matrices, which is the usual criterion for compatible observables. In general, every eigenvector for X' which is not orthogonal to ψ coincides with an eigenvector for X.

We close this section with a consideration of the expectation of a function relative to a measurement. Suppose we want to measure a function $g:\Omega\to\mathbb{R}$ using a measurement X. We first define the amplitude of g when X results in x by

$$f_X(g)(x)=\sum\{g(\omega)f(\omega): X(\omega)=x\} \tag{2.13}$$

Notice that (2.13) is the sum of the values of g times the amplitudes of these values along $X^{-1}(x)$, so it is similar to a probability average. In this sense, we can think of (2.13) as an amplitude average. Also, (2.13) is a generalization of (2.4), since if $g=\chi_A$, then $f_X(g)(x)=f_X(A)(x)$. We define the (X,f)-*pseudoexpectation* of g by

$$E_{X,f}(g)=\sum_{x\in R(X)}f_X(g)(x)\overline{f_X(x)} \tag{2.14}$$

where $\overline{f_X(x)}$ is the complex conjugate of $f_X(x)$. Notice that (2.14) is a generalization of $P_{X,f}(B)$, since if $g=\chi_{X^{-1}(B)}$, then $E_{X,f}(g)=P_{X,f}(B)$. Equation (2.13) can be used to show the correspondence between measurements and functions on Ω with linear operators on H_X. In fact, the map $f_X\mapsto f_X(g)$ takes elements of H_X to elements of H_X. Moreover, if we extend

(2.13) to include unnormalized amplitudes, then this map becomes linear. Therefore, corresponding to g we obtain a linear operator $\hat{g}$ on H_X satisfying

$$\hat{g}f_X = f_X(g) \tag{2.15}$$

Then (2.14) becomes

$$E_{X,f}(g) = \sum_{x \in R(X)} (\hat{g}f_X)(x)\overline{f_X(x)} = \langle \hat{g}f_X, f_X \rangle \tag{2.16}$$

which is the usual quantum mechanical formula. In particular, suppose $R(X) \subseteq \mathbb{R}$. Then (2.13) gives

$$f_X(X)(x) = x \sum \{f(\omega): X(\omega) = x\} = xf_X(x)$$

Hence, X is represented by the operator $\tilde{X}$, where $\tilde{X}f_X(x) = xf_X(x)$. This gives the usual representation Hilbert space H_X in which X is diagonal.

3. HORIZONTAL SUMS

This section illustrates some of the concepts of Section 2 and also begins our study of combined systems. To conform with probabilistic terminology, we call the set Ω a *sample space*. Let Ω_1 and Ω_2 be finite sample spaces and let $\Omega = \Omega_1 \times \Omega_2$ be the Cartesian product of Ω_1 and Ω_2,

$$\Omega = \{(\omega_1, \omega_2): \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$$

If X_1, X_2 are measurements on Ω_1, Ω_2, respectively, we define the measurements $\hat{X}_1, \hat{X}_2$ on Ω by $\hat{X}_1(\omega_1, \omega_2) = X_1(\omega_1)$, $\hat{X}_2(\omega_1, \omega_2) = X_2(\omega_2)$. If $\mathscr{A}_1$ and $\mathscr{A}_2$ are catalogs on Ω_1, Ω_2, respectively, we define the *horizontal sum* of $\mathscr{A}_1$ and $\mathscr{A}_2$ as

$$\mathscr{A}_1 + \mathscr{A}_2 = \{\hat{X}_1, \hat{X}_2: X_1 \in \mathscr{A}_1, X_2 \in \mathscr{A}_2\}$$

The extension of this definition to any number of summands is straightforward. The next example illustrates the importance of this construction.

Example 1 (Spin Chain). Let $\Omega_1 = \{u, d\}$ be the sample space describing the spin in the z direction of a spin-1/2 particle and let X be the spin measurement given by $X(u) = 1/2$, $X(d) = -1/2$. Suppose a spin-1/2 particle initially has spin up and we then perform spin measurements in the z direction at one time unit and at two time units. Letting $\Omega_0 = \{u\}$, this can be described by the sample space

$$\Omega = \Omega_0 \times \Omega_1 \times \Omega_1 = \{(\omega_1, \omega_2, \omega_2): \omega_0 = u, \omega_1, \omega_2 \in \Omega_1\}$$

We then define the spin measurements $\hat{X}_0, \hat{X}_1, \hat{X}_2$ on Ω by $\hat{X}_j(\omega_0, \omega_1, \omega_2) = X(\omega_j)$, $j = 0, 1, 2$. For $\omega \in \Omega$, let $n(\omega)$ be the number of successive spin changes; for example, $n((u, u, d))=1$, $n((u, d, u))=2$. Define $f:\Omega\to\mathbb{C}$ by $f(\omega)=i^{n(\omega)}/2$, $i=\sqrt{-1}$. Then

$$f_{\hat{X}_0}\left(\frac{1}{2}\right)=\frac{1}{2}\sum_{\omega\in\Omega} i^{n(\omega)}=\frac{1}{2}(1+2i+i^2)=i$$

$$f_{\hat{X}_1}\left(\frac{1}{2}\right)=\frac{1}{2}(1+i),\qquad f_{\hat{X}_1}\left(-\frac{1}{2}\right)=\frac{1}{2}(-1+i)$$

$$f_{\hat{X}_2}\left(\frac{1}{2}\right)=0,\qquad f_{\hat{X}_1}\left(-\frac{1}{2}\right)=i$$

It follows that f is an amplitude density for $\hat{X}_0$, $\hat{X}_1$, and $\hat{X}_2$. We have

$$f_{\hat{X}_1}\left[\hat{X}_2^{-1}\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)=f(u,u,u)=\frac{1}{2}$$

$$f_{\hat{X}_1}\left[\hat{X}_2^{-1}\left(\frac{1}{2}\right)\right]\left(-\frac{1}{2}\right)=f(u,d,u)=-\frac{1}{2}$$

$$f_{\hat{X}_1}\left[\hat{X}_2^{-1}\left(-\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)=f(u,u,d)=\frac{i}{2}$$

$$f_{\hat{X}_1}\left[\hat{X}_2^{-1}\left(-\frac{1}{2}\right)\right]\left(-\frac{1}{2}\right)=f(u,d,d)=\frac{i}{2}$$

Hence,

$$P_{\hat{X}_1,f}\left[\hat{X}_2^{-1}\left(\frac{1}{2}\right)\right]=\frac{1}{2}\neq 0=P_{\hat{X}_2,f}\left(\frac{1}{2}\right)$$

$$P_{\hat{X}_1,f}\left[\hat{X}_2^{-1}\left(-\frac{1}{2}\right)\right]=\frac{1}{2}\neq 1=P_{\hat{X}_2,f}\left(-\frac{1}{2}\right)$$

so a spin measurement at time 1 interferes with a spin measurement at time 2. Moreover,

$$f_{\hat{X}_2}\left[\hat{X}_1^{-1}\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)=\frac{1}{2},\qquad f_{\hat{X}_2}\left[\hat{X}_1^{-1}\left(\frac{1}{2}\right)\right]\left(-\frac{1}{2}\right)=\frac{i}{2}$$

$$f_{\hat{X}_2}\left[\hat{X}_1^{-1}\left(-\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)=-\frac{1}{2},\qquad f_{\hat{X}_2}\left[\hat{X}_1^{-1}\left(-\frac{1}{2}\right)\right]\left(-\frac{1}{2}\right)=\frac{i}{2}$$

Hence,

$$P_{\hat{X}_2,f}\left[\hat{X}_1^{-1}\left(\frac{1}{2}\right)\right]=\frac{1}{2}=P_{\hat{X}_1,f}\left(\frac{1}{2}\right)$$

$$P_{\hat{X}_2,f}\left[\hat{X}_1^{-1}\left(-\frac{1}{2}\right)\right]=\frac{1}{2}=P_{\hat{X}_1,f}\left(-\frac{1}{2}\right)$$

so a spin measurement at time 2 does not interfere with a spin measurement at time 1. This shows that noninterference is not a symmetric relation.

One can make a similar analysis for longer spin-1/2 chains. If measurements are performed at times $0, 1, 2, \ldots, m$, we would construct the sample space $\Omega_0 \times \Omega_1 \times \cdots \times \Omega_1$, where Ω_1 is repeated m times. In this case, we would define $f(\omega) = i^{n(\omega)}/2^{m/2}$. It can then be shown that f is an amplitude density for the measurements $\hat{X}_j, j = 0, \ldots, m$. Notice that in all these cases

$$\left|\sum_{\omega \in \Omega} f(\omega)\right| = \left|f_{\hat{X}_0}\left(\frac{1}{2}\right)\right| = 1 \tag{3.1}$$

Other repeated measurements such as higher spin measurements can be described in this way. Of course, in such cases the definition of f would be more complicated. ■

Let Ω_1, Ω_2 be finite sample spaces, $\mathscr{A}_1$, $\mathscr{A}_2$ catalogs on Ω_1, Ω_2, respectively, and $f_1 \in \mathscr{D}(\mathscr{A}_1), f_2 \in \mathscr{D}(\mathscr{A}_2)$. We now give a method for combining f_1 and f_2 to form an amplitude density in $\mathscr{D}(\mathscr{A}_1 + \mathscr{A}_2)$. Suppose f_1, f_2 satisfy

$$\left|\sum_{\omega_1 \in \Omega_1} f_1(\omega_1)\right| = \left|\sum_{\omega_2 \in \Omega_2} f_2(\omega_2)\right| = c \neq 0 \tag{3.2}$$

Although (3.2) is a fairly strong restriction, it does hold in various situations. For example, if $\Omega_1 = \Omega_2$ and $f_1 = f_2$, then (3.2) certainly holds. Also, by (3.1), amplitude densities for the spin chains in Example 1 satisfy (3.2) with $c = 1$. We define $f_1 \circ f_2 : \Omega_1 \times \Omega_2 \to \mathbb{C}$ by

$$(f_1 \circ f_2)(\omega_1, \omega_2) = \frac{f_1(\omega_1) f_2(\omega_2)}{c}$$

Lemma 3.1. If $f_1 \in \mathscr{D}(\mathscr{A}_1)$, $f_2 \in \mathscr{D}(\mathscr{A}_2)$ satisfy (3.2), then $f_1 \circ f_2 \in \mathscr{D}(\mathscr{A}_1 + \mathscr{A}_2)$.

Proof. Let $X_1 \in \mathscr{A}_1$, $X_2 \in \mathscr{A}_2$. We must show that $f_1 \circ f_2$ is an amplitude density for $\hat{X}_1$ and $\hat{X}_2$; that is, the normalization condition (2.3) holds. From (3.2), letting

$$\sum_{\omega_1 \in \Omega_1} f_1(\omega_1) = c_1, \qquad \sum_{\omega_2 \in \Omega_2} f_2(\omega_2) = c_2$$

we have $|c_1| = |c_2| = c$. For $x_1 \in R(\hat{X}_1)$ we have

$$\begin{aligned}(f_1 \circ f_2)_{\hat{X}_1}(x_1) &= \sum \{f_1 \circ f_2(\omega_1, \omega_2) : \hat{X}_1(\omega_1, \omega_2) = x_1\} \\ &= \frac{1}{c} \sum \{f_1(\omega_1) f_2(\omega_2) : X_1(\omega_1) = x_1, \omega_2 \in \Omega_2\} \\ &= \frac{1}{c} \sum_{\omega_2 \in \Omega_2} f_2(\omega_2) \sum \{f_1(\omega_1) : X_1(\omega_1) = x_1\} \\ &= \frac{c_2}{c} f_{1X_1}(x_1) \end{aligned} \tag{3.3}$$

Hence,

$$\sum_{x_1 \in R(\hat{X}_1)} |(f_1 \circ f_2)_{\hat{X}_1}(x_1)|^2 = \sum_{x_1 \in R(X_1)} |f_{1X_1}(x_1)|^2 = 1$$

Similarly, if $x_2 \in R(\hat{X}_2)$, then

$$(f_1 \circ f_2)_{\hat{X}_2}(x_2) = \frac{c_1}{c} f_{2X_2}(x_2)$$

and the normalization condition again holds. ■

We now study the independence of $\hat{X}_1$ and $\hat{X}_2$ relative to $f = f_1 \circ f_2$. Letting $A \subseteq R(\hat{X}_2)$ and $x_1 \in R(\hat{X}_1)$, we have

$$\begin{aligned} f_{\hat{X}_1}[\hat{X}_2^{-1}(A)](x_1) &= \sum \{f(\omega_1, \omega_2): \hat{X}_1(\omega_1, \omega_2) = x_1, \hat{X}_2(\omega_1, \omega_2) \in A\} \\ &= \frac{1}{c} \sum \{f_1(\omega_1) f_2(\omega_2): X_1(\omega_1) = x_1, X_2(\omega_2) \in A\} \\ &= \frac{1}{c} f_{1X_1}(x_1) \sum \{f_2(\omega_2): X_2(\omega_2) \in A\} \end{aligned} \quad (3.4)$$

Hence,

$$\begin{aligned} P_{\hat{X}_1, f}[\hat{X}_2^{-1}(A)] &= \sum_{x_1 \in R(\hat{X}_1)} |f_{\hat{X}_1}[\hat{X}_2^{-1}(A)](x_1)|^2 \\ &= \frac{1}{c^2} |\sum \{f_2(\omega_2): X_2(\omega_2) \in A\}|^2 \end{aligned} \quad (3.5)$$

Now let $B \subseteq R(\hat{X}_1)$. Applying (3.3)-(3.5), we have

$$\begin{aligned} \sum_{x_1 \in B} |f_{\hat{X}_1}[\hat{X}_2^{-1}(A)](x_1)|^2 &= \frac{1}{c^2} \sum_{x_1 \in B} |f_{1X_1}(x_1)|^2 |\sum \{f_2(\omega_2): X_2(\omega_2) \in A\}|^2 \\ &= P_{\hat{X}_1, f}(B) P_{\hat{X}_1, f}[\hat{X}_2^{-1}(A)] \end{aligned}$$

We conclude from (2.10) that $\hat{X}_1$ and $\hat{X}_2$ are independent relative to f. In this sense, any $X_1 \in \mathscr{A}_1$ is independent of any $X_2 \in \mathscr{A}_2$ relative to f. This is not surprising, since f has a product form.

We now consider interference. If follows from (3.5) that

$$P_{\hat{X}_1, f}[\hat{X}_2^{-1}(A)] = P_{\hat{Y}_1, f}[\hat{X}_2^{-1}(A)]$$

for any X_1, $Y_1 \in \mathscr{A}_1$ and $X_2 \in \mathscr{A}_2$. Moreover, it is easily shown that

$$P_{\hat{X}_2, f}(A) = \sum_{x_2 \in A} |f_{2X_2}(x_2)|^2 \quad (3.6)$$

Now (3.5) and (3.6) certianly look different. In fact, we shall see in the next example that there can exist an $f_2 \in \mathscr{D}(\mathscr{A}_2)$ for which they are different.

Hence, for such an $f=f_1\circ f_2$, every $\hat{X}_1$ interferes with every $\hat{X}_2$. This is related to the *EPR* problem. The measurements $\hat{X}_1$ and $\hat{X}_2$ are separated in the sense that they cannot communicate, since they act on different parts of the system. In fact, they are statistically independent. However, the amplitude density $f=f_1\circ f_2$ produces nonlocal communication resulting in interference. This also shows that there are independent measurements that interfere.

Example 2. Let $\Omega=\{\omega_1, \omega_2, \omega_3\}$ and let X_2 be a measurement given by $X_2(\omega_j)=\lambda_j, j=1, 2, 3$, where the λ_j are distinct. Define $f_2:\Omega\to\mathbb{C}$ by $f_2(\omega_1)=1/3+i/\sqrt{3}$, $f_2(\omega_2)=1/3-i/\sqrt{3}$, $f_2(\omega_3)=1/3$. Then

$$\left|\sum_{\omega\in\Omega} f_2(\omega)\right|=c=1$$

and $f_{X_2}(\lambda_j)=f_2(\omega_j)$, $j=1, 2, 3$. Hence,

$$\sum_{\lambda\in R(X_2)} |f_{X_2}(\lambda)|^2=1$$

so the normalization condition holds. Letting $A=\{\lambda_1, \lambda_2\}\subseteq R(X_2)$, we have

$$|\sum\{f_2(\omega): X_2(\omega)\in A\}|^2=\frac{4}{9}\neq\frac{8}{9}=\sum_{x_2\in A} |f_{2X_2}(x_2)|^2$$

We conclude that (3.5) and (3.6) do not agree, in general. ■

Let $\mathscr{A}=\mathscr{A}_1+\mathscr{A}_2$. We now characterize those $f\in\mathscr{D}(\mathscr{A})$ that have the form $f=f_1\circ f_2$, $f_1\in\mathscr{D}(\mathscr{A}_1)$, $f_2\in\mathscr{D}(\mathscr{A}_2)$. We say that $f\in\mathscr{D}(\mathscr{A})$ is *factorizable* if the following conditions hold:

(a) $\sum\{f(\omega_1, \omega_2): \omega_1\in\Omega_1, \omega_2\in\Omega_2\}=d\neq 0$.

(b) If $f'(\omega_1)=\sum_{\omega_2\in\Omega_2} f(\omega_1, \omega_2)$, then $f'(\omega_1)=0$ implies $f(\omega_1, \omega_2)=0$ for every $\omega_2\in\Omega_2$.

(c) $f(\omega_1, \omega_2)/f'(\omega_1)$ depends only on ω_2 whenever $f'(\omega_1)\neq 0$.

Theorem 3.2. Let $f\in\mathscr{D}(\mathscr{A})$. Then there exist $f_1\in\mathscr{D}(\mathscr{A}_1)$, $f_2\in\mathscr{D}(\mathscr{A}_2)$ such that $f=f_1\circ f_2$ if and only if f is factorizable.

Proof. Suppose $f=f_1\circ f_2$. Then, defining c_1, c_2 as in the proof of Lemma 3.1, we have

$$\sum\{f(\omega_1, \omega_2): \omega_1\in\Omega_1, \omega_2\in\Omega_2\}=\frac{1}{c}\sum_{\omega_1\in\Omega_1} f_1(\omega_1)\sum_{\omega_2\in\Omega_2} f_2(\omega_2)$$

$$=\frac{c_1c_2}{c}\neq 0$$

so (a) holds. Since $f'(\omega_1)=c_2 f_1(\omega_1)/c$, if $f'(\omega_1)=0$, then $f_1(\omega_1)=0$, so $f(\omega_1,\omega_2)=0$ for every $\omega_2\in\Omega_2$. Hence, (b) holds. If $f'(\omega_1)\neq 0$, then

$$\frac{f(\omega_1,\omega_2)}{f'(\omega_1)}=\frac{1}{c_2}f_2(\omega_2)$$

is a function of only ω_2, so (c) holds. Conversely, suppose $f\in\mathscr{D}(\mathscr{A})$ is factorizable. We first show that $f'\in\mathscr{D}(\mathscr{A}_1)$. For $X_1\in\mathscr{A}$ and $x_1\in R(X_1)$ we have

$$\begin{aligned}f'_{X_1}(x_1)&=\sum\{f'(\omega_1)\colon X_1(\omega_1)=x_1\}\\&=\sum_{\omega_1\in X_1^{-1}(x_1)}\sum_{\omega_2\in\Omega_2} f(\omega_1,\omega_2)\\&=\sum\{f(\omega_1,\omega_2)\colon \hat{X}_1(\omega_1,\omega_2)=x_1\}=f_{\hat{X}_1}(x_1)\end{aligned}\tag{3.7}$$

Hence,

$$\sum_{x_1\in R(X_1)}|f'_{X_1}(x_1)|^2=\sum_{x_1\in R(\hat{X}_1)}|f_{\hat{X}_1}(x_1)|^2=1$$

so $f'\in\mathscr{D}(\mathscr{A}_1)$. Since $f'\neq 0$, there exists an ω_1 such that $f'(\omega_1)\neq 0$. Define

$$f''(\omega_2)=\frac{|d|f(\omega_1,\omega_2)}{f'(\omega_1)}$$

for every $\omega_1\in\Omega_2$. By (c) we have

$$\frac{1}{|d|}f'(\omega_1)f''(\omega_2)=f(\omega_1,\omega_2)$$

for every $\omega_1\in\Omega_1$, $\omega_2\in\Omega_2$. Applying (a) and (b) gives

$$f''(\omega_2)=\frac{|d|}{d}\sum_{\omega_1\in\Omega_1} f(\omega_1,\omega_2)$$

As in (3.7), if $X_2\in\mathscr{A}_2$ and $x_2\in R(X_2)$, we have

$$f''_{X_2}(x_2)=\frac{|d|}{d}f_{\hat{X}_2}(x_2)$$

so $f''\in\mathscr{D}(\mathscr{A}_2)$. Since

$$\left|\sum_{\omega_1\in\Omega_1} f'(\omega_1)\right|=\left|\sum_{\omega_2\in\Omega_2} f''(\omega_2)\right|=|d|$$

we have $f=f'\circ f''$. ■

4. MATHEMATICAL FORMULATION

We now summarize the full mathematical theory motivated by the considerations in Section 2. This theory has already been developed in Gudder (1988*a*,*b*, 1989, and 1990) and we refer the reader to these references for more details.

Let Ω be a nonempty set which we call a *sample space* and whose elements we call *sample points.* A map $X:\Omega\to R(X)$ is a *measurement* if the following conditions hold.

(M1) $R(X)$ is the base space of a measure space $(R(X), \Sigma_X, \mu_X)$.

(M2) For every $x\in R(X)$, $X^{-1}(x)$ is the base space of a measure space $(X^{-1}(x), \Sigma_X^x, \mu_X^x)$.

We call the elements of $R(X)$, *X-outcomes,* the sets in Σ_X, *X-events,* and $X^{-1}(x)$ the *fiber* (or *sample*) *over* x. Notice that $\mathscr{E}(X)=\{X^{-1}(B)\colon B\in\Sigma_X\}$ is a σ-algebra of subsets of Ω. We call the sets in $\mathscr{E}(X)$, *X-sample events.*

A function $f:\Omega\to\mathbb{C}$ is an *amplitude density* for the measurement X if the following conditions hold:

(A1) $f|X^{-1}(x)\in L^1(X^{-1}(x), \Sigma_X^x, \mu_X^x)$ for all $x\in R(X)$.
(A2) $f_X(x)\equiv\int f\,d\mu_X^x\in L^2(R(X), \Sigma_X, \mu_X)\equiv H_X$.
(A3) $\|f_X\|^2=\int|f_X|^2\,d\mu_X=1$.

We call H_X the *Hilbert space* for X and f_X the *(X,f)-wave function.* Of course, (A2) and (A3) correspond to (2.1) and (2.3), respectively, in the finite case.

A subset $A\subseteq\Omega$ is an *(X,f)-sample event* if the following conditions hold:

(S1) $A\cap X^{-1}(x)\in\Sigma_X^x$ for every $x\in R(X)$.
(S2) $f_X(A)(x)\equiv\int_{A\cap X^{-1}(x)}f\,d\mu_X^x\in H_X$.

Denoting the set of (X,f)-sample events by $\mathscr{E}(X,f)$, it is clear that $\mathscr{E}(X)\subseteq\mathscr{E}(X,f)$. If $A\in\mathscr{E}(X,f)$, the *(X,f)-pseudoprobability* of A is

$$P_{X,f}(A)=\int|f_X(A)|^2\,d\mu_X=\|f_X(A)\|^2 \tag{4.1}$$

Notice that (S2) and (4.1) correspond to (2.4) and (2.6), respectively. For $B\in\Sigma_X$, as in the finite case, it is easy to show that

$$P_{X,f}(B)\equiv P_{X,f}[X^{-1}(B)]=\int_B|f_X|^2\,d\mu_X$$

If $A\in\mathscr{E}(X,f)$, $B\in\Sigma_X$, it can be shown that $X^{-1}(B)\cap A\in\mathscr{E}(X,f)$ (Gudder, 1988*a* and 1990). If $P_{X,f}(A)\neq 0$, we define the *conditional* *probability* $P_{X,f}(B|A)$ as in (2.7) and it follows as in (2.9) that

$$P_{X,f}(B|A) = \frac{1}{P_{X,f}(A)} \int_B |f_X(A)|^2 \, d\mu_X$$

We say that X is *f-independent* of a measurement Y if $\mathscr{E}(Y) \subseteq \mathscr{E}(X,f)$ and

$$\int_B |f_X[Y^{-1}(A)]|^2 \, d\mu_X = P_{X,f}(B) P_{X,f}[Y^{-1}(A)]$$

for every $B \in \Sigma_X$, $A \in \Sigma_Y$. As before, this reduces to

$$P_{X,f}(B|Y^{-1}(A)) = P_{X,f}(B)$$

for every $B \in \Sigma_X$, $A \in \Sigma_Y$ with $P_{X,f}[Y^{-1}(A)] \neq 0$. Also, *X does not interfere with Y relative to f* if $\mathscr{E}(Y) \subseteq \mathscr{E}(X,f)$ and for every $A \in \Sigma_Y$ we have

$$P_{X,f}[Y^{-1}(A)] = P_{Y,f}(A)$$

We denote by $L^2(X,f)$ the functions $g: \Omega \to \mathbb{R}$ satisfying the following conditions:

(X1) $gf|X^{-1}(x) \in L^1(X^{-1}(x), \Sigma_X^x, \mu_X^x)$ for all $x \in R(X)$.
(X2) $f_X(g)(x) \equiv \int gf \, d\mu_X^x \in H_X$.

We call $f_X(g)$ the *(X,f)-amplitude density* of g and notice that (X2) corresponds to (2.13). Corresponding to (2.14) we define the (X,f)-*pseudoexpectation* of $g \in L^2(X,f)$ by

$$E_{X,f}(g) = \int f_X(g) \bar{f}_X \, d\mu_X = \langle f_X(g), f_X \rangle$$

Until now we have only considered a single amplitude density f. However, it is frequently important to consider several amplitudes simultaneously; in particular, linear combinations of amplitudes. Denote by $\hat{\mathscr{A}}(\Omega)$ the set of all measurements on Ω. A nonempty subset $\mathscr{A} \subseteq \hat{\mathscr{A}}(\Omega)$ is called a *catalog*. A function $f: \Omega \to \mathbb{C}$ is an *amplitude* for a catalog $\mathscr{A}$ if f satisfies (A1), (A2) for every $X \in \mathscr{A}$ and the following condition:

(A3′) $\|f_X\| = \|f_Y\|$ for every X, $Y \in \mathscr{A}$.

We denote the set of amplitudes for $\mathscr{A}$ by $\mathscr{H}(\mathscr{A})$ and call $\mathscr{H}(\mathscr{A})$ the *amplitude space* for $\mathscr{A}$. If $f \in \mathscr{H}(\mathscr{A})$, we write $\|f\| = \|f_X\|$, where $X \in \mathscr{A}$ is arbitrary. Of course, if $\|f\| = 1$, then f is an amplitude density. We denote the set of amplitude densities for $\mathscr{A}$ by $\mathscr{D}(\mathscr{A})$. Notice that if $f \in \mathscr{H}(\mathscr{A})$, $a \in \mathbb{C}$, then $af \in \mathscr{H}(\mathscr{A})$ and $\|af\| = |a| \, \|f\|$. Also, if $f \in \mathscr{H}(\mathscr{A})$ with $\|f\| \neq 0$, then $f/\|f\| \in \mathscr{D}(\mathscr{A})$.

For f, $g \in \mathscr{H}(\mathscr{A})$ we write $f s g$ if for every X, $Y \in \mathscr{A}$ we have

$$\int f_X \bar{g}_X \, d\mu_X = \int f_Y \bar{g}_Y \, d\mu_Y \tag{4.2}$$

If (4.2) holds, we denote this expression by $\langle f, g\rangle$. Notice that s is a reflexive, symmetric relation and if $f s g$, then $af s g$ for all $a \in \mathbb{C}$. We call s the *superposition relation.* The following result is proved in Gudder (1988*a* and 1990).

Theorem 4.1. For $f, g \in \mathscr{H}(\mathscr{A})$, $f s g$ if and only if $af + bg \in \mathscr{H}(\mathscr{A})$ for every $a, b \in \mathbb{C}$.

For $B \subseteq \mathscr{H}(\mathscr{A})$ we write

$$B^s = \{f \in \mathscr{H}(\mathscr{A}) : f s g \text{ for all } g \in B\}$$

We call $B \subseteq \mathscr{H}(\mathscr{A})$ an *s-set* if $B \subseteq B^s$. Thus, B is an s-set if and only if $f s g$ for all $f, g \in B$. It is clear that singleton sets are s-sets and hence every $f \in \mathscr{H}(\mathscr{A})$ is in an s-set. Moreover, by Zorn's lemma, every s-set is contained in a maximal s-set. We denote the collection of maximal s-sets by $\mathscr{M}(\mathscr{A})$. Elements of $\mathscr{M}(\mathscr{A})$ are maximal sets of amplitudes for which superpositions are allowed. They correspond to superselection sectors for a physical system. It follows from Theorem 4.1 that if $M \in \mathscr{M}(\mathscr{A})$, then M is closed under addition and scalar multiplication, so M is a linear space. We call $f \in \mathscr{H}(\mathscr{A})$ a *null amplitude* if $\|f\| = 0$. It is clear that the set of null amplitudes forms a linear subspace of every $M \in \mathscr{M}(\mathscr{A})$ If we identify amplitudes that differ by a null amplitude, it is straightforward to show that $\langle \cdot, \cdot \rangle$ is an inner product on M. We then call this inner product space a *sector.* The collection of all sectors is denoted $\mathscr{S}(\mathscr{A})$. In general, $\mathscr{A}$ can have many sectors (Gudder, in preparation).

To illustrate the power of this formulation of quantum probability, we now show that it includes traditional nonrelativistic quantum mechanics as a special case. We have previously shown this for the catalog $\{Q, P\}$, where Q and P are the position and momentum measurements, respectively (Gudder, 1988*a*). We now give a much simpler argument using the catalog $\mathscr{A} = \{Q\}$. For simplicity we consider the one-dimensional case, but these results easily generalize to three dimensions.

We take as our sample space the two-dimensional phase space

$$\Omega = \{(q, p) : q, p \in \mathbb{R}\}$$

and define the *position measurement* $Q : \Omega \to \mathbb{R}$ by $Q(q, p) = q$. We place Lebesgue measure on the fibers and range of Q so Q is indeed a measurement. We define the *momentum function* $P : \Omega \to \mathbb{R}$ by $P(q, p) = p$. Since we are considering the catalog $\mathscr{A} = \{Q\}$, P is treated as a function and not as a measurement. Then H_Q becomes the usual position-representation Hilbert space. Since we want to describe dynamics, our amplitude densities will be functions of time.

Let $\psi(q, t)$ be a complex-valued function which is twice differentiable with respect to q and differentiable with respect to t. Moreover, we assume that ψ, $\partial\psi/\partial q$, $\partial^2\psi/\partial q^2 \in L^2(\mathbb{R}, dq)$ and $\|\psi\| = 1$. For each $t \in \mathbb{R}$, define the function $f: \Omega \to \mathbb{C}$ by

$$f(q, p, t) = \frac{1}{(2\pi\hbar)^{1/2}} \hat{\psi}(p, t)\, e^{iqp/\hbar}$$

where $\hat{\psi}$ is the Fourier transform of ψ. The Q-wave function becomes

$$f_Q(q, t) = \int f(q, p, t)\, dp = \frac{1}{(2\pi\hbar)^{1/2}} \int \hat{\psi}(p, t)\, e^{iqp/\hbar}\, dp = \psi(q, t)$$

It follows that $f_Q \in L^2(\mathbb{R}, dq) = H_Q$ and $\|f_Q\| = 1$, so $f \in \mathscr{D}(\mathscr{A})$ for each $t \in \mathbb{R}$. Moreover, $P \in L^2(Q, f)$ [see Conditions (X1), (X2)] since

$$\begin{aligned} f_Q(P)(q, t) &= \int pf(q, p, t)\, dp \\ &= \frac{1}{(2\pi\hbar)^{1/2}} \int p\hat{\psi}(p, t)\, e^{idqp/\hbar}\, dp \\ &= \frac{1}{(2\pi\hbar)^{1/2}} \left(-i\hbar \frac{\partial}{\partial q}\right) \int \hat{\psi}(p, t)\, e^{iqp/\hbar}\, dp \\ &= -i\hbar \frac{\partial\psi}{\partial q}(q, t) \end{aligned}$$

Similarly, $f_Q(P^2) = -\hbar^2\partial^2\psi/\partial q^2$ and if ψ is sufficiently smooth, $f_Q(P^n) = [-i\hbar\, \partial/\partial q]^n\psi$. More generally, if ψ is sufficiently smooth, then for any polynomial g we have $f_Q(g(P)) = g[-i\hbar\, \partial/\partial q]\psi$. Moreover, if $V: \mathbb{R} \to \mathbb{R}$ is any function such that $V(q)\psi(q, t) \in H_Q$, then

$$f_Q(V(Q))(q, t) = \int V(q) f(q, p, t)\, dp = V(q)\psi(q, t)$$

We conclude that $g(P)$ is represented by the operator $g[-i\hbar\, \partial/\partial q]$ and $V(Q)$ by the operator which multiplies by $V(q)$. In this way we have derived the Bohr correspondence principle.

We now derive the Schrödinger equation from Hamilton's equation $dp/dt = -\partial H/\partial q$. Suppose that the Hamiltonian has the form

$$H(q, p) = \frac{p^2}{2m} + V(q)$$

We now assume that Hamilton's equation holds in the amplitude average. Hence,

$$\frac{d}{dt} \int pf(q, p, t)\, dp = -\frac{\partial}{\partial q} \int H(q, p) f(q, p, t)\, dp$$

It follows that

$$\frac{d}{dt}\left(-i\hbar\frac{\partial\psi}{\partial q}\right)=-\frac{\partial}{\partial q}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial q^2}+V(q)\psi\right]$$

Interchanging the order of differentiation on the left side of this equation and integrating with respect to q now gives Schrödinger's equation

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial q^2}+V(q)\psi$$

5. DIRECT SUMS

Let $\mathcal{A}_1\subseteq\hat{\mathcal{A}}(\Omega_1)$, $\mathcal{A}_2\subseteq\hat{\mathcal{A}}(\Omega_2)$ be catalogs on the sample spaces Ω_1, Ω_2. Form the sample space consisting of the disjoint union $\Omega=\Omega_1\dot{\cup}\Omega_2$ of Ω_1 and Ω_2. For $X_1\in\mathcal{A}_1$, $X_2\in\mathcal{A}_2$ define $X=X_1\oplus X_2$ as the function

$$X:\quad \Omega\to R(X_1)\dot{\cup} R(X_2)$$

given by

$$X(\omega)=\begin{cases}X_1(\omega) & \text{if} \quad \omega\in\Omega_1\\ X_2(\omega) & \text{if} \quad \omega\in\Omega_2\end{cases}$$

We make X into a measurement by defining

$$\Sigma_X=\{A_1\dot{\cup}A_2\colon A_1\in\Sigma_{X_1}, A_2\in\Sigma_{X_2}\}$$

and

$$\mu_X(A_1\dot{\cup}A_2)=\mu_{X_1}(A_1)+\mu_{X_2}(A_2)$$

Moreover, on the fiber $X^{-1}(x)$ we define

$$\Sigma_X^x=\begin{cases}\Sigma_{X_1}^x & \text{if} \quad x\in R(X_1)\\ \Sigma_{X_2}^x & \text{if} \quad x\in R(X_2)\end{cases}$$

and for $A\in\Sigma_X^x$ define

$$\mu_X^x(A)=\begin{cases}\mu_{X_1}^x(A) & \text{if} \quad A\in\Sigma_{X_1}^x\\ \mu_{X_2}^x(A) & \text{if} \quad A\in\Sigma_{X_2}^x\end{cases}$$

We define the *direct sum* of $\mathcal{A}_1, \mathcal{A}_2$ to be

$$\mathcal{A}=\mathcal{A}_1\oplus\mathcal{A}_2=\{X_1\oplus X_2\colon X_1\in\mathcal{A}_1, X_2\in\mathcal{A}_2\}$$

Then $\mathcal{A}_1\oplus\mathcal{A}_2$ is a catalog in $\hat{\mathcal{A}}(\Omega_1\dot{\cup}\Omega_2)$.

Let $f_1\in\mathcal{H}(\mathcal{A}_1)$, $f_2\in\mathcal{H}(\mathcal{A}_2)$ and define $f=f_1\oplus f_2\colon\Omega\to\mathbb{C}$ by

$$f(\omega)=\begin{cases}f_1(\omega) & \text{if} \quad \omega\in\Omega_1\\ f_2(\omega) & \text{if} \quad \omega\in\Omega_2\end{cases}$$

For $X = X_1 \oplus X_2$ we have $R(X) = R(X_1) \dot{\cup} R(X_2)$. Let $x \in R(X)$ and suppose $x \in R(X_1)$. Then $f|X^{-1}(x) = f_1|X_1^{-1}(x)$. Similarly, if $x \in R(X_2)$, then $f|X^{-1}(x) = f_2|X_2^{-1}(x)$. Hence,

$$f_X(x) = \begin{cases} f_{1X_1}(x) & \text{if} \quad x \in R(X_1) \\ f_{2X_2}(x) & \text{if} \quad x \in R(X_2) \end{cases}$$

It follows that

$$\int_{R(X)} |f_X|^2 \, d\mu_X = \int_{R(X_1)} |f_{1X_1}|^2 \, d\mu_{X_1} + \int_{R(X_2)} |f_{2X_2}|^2 \, d\mu_{X_2}$$
$$= \|f_1\|^2 + \|f_2\|^2$$

We conclude that $f \in \mathscr{H}(\mathscr{A})$ and $\|f\|^2 = \|f_1\|^2 + \|f_2\|^2$. Moreover, if $f_1 \in \mathscr{D}(\mathscr{A}_1)$, $f_2 \in \mathscr{D}(\mathscr{A}_2)$ and $a, b \in \mathbb{C}$ satisfy $|a|^2 + |b|^2 = 1$, then $af_1 \oplus bf_2 \in \mathscr{D}(\mathscr{A})$.

We now discuss interference and independence. Let $X = X_1 \oplus X_2 \in \mathscr{A}_1 \oplus \mathscr{A}_2$, $f = af_1 \oplus bf_2$, $f_i \in \mathscr{D}(\mathscr{A}_i)$, $i = 1, 2$, $|a|^2 + |b|^2 = 1$ and let $A = A_1 \dot{\cup} A_2 \in \mathscr{E}(X, f)$. It follows that $A_1 \in \mathscr{E}(X_1, f_1)$ and $A_2 \in \mathscr{E}(X_2, f_2)$. If $x \in R(X_1)$, then

$$f_X(A)(x) = \int_{A \cap X^{-1}(x)} f \, d\mu_X^x = a \int_{A_1 \cap X_1^{-1}(x)} f_1 \, d\mu_{X_1}^x = af_{1X_1}(A_1)(x)$$

Similarly, if $x \in R(X_2)$, then $f_X(A)(x) = bf_{2X_2}(A_2)(x)$. Hence,

$$P_{X,f}(A) = \int_{R(X)} |f_X(A)|^2 \, d\mu_X$$
$$= |a|^2 \int_{R(X_1)} |f_{1X_1}(A_1)|^2 \, d\mu_{X_1} + |b|^2 \int_{R(X_2)} |f_{2X_2}(A_2)|^2 \, d\mu_{X_2}$$
$$= |a|^2 P_{X_1, f_1}(A_1) + |b|^2 P_{X_2, f_2}(A_2) \tag{5.1}$$

In particular, (5.1) shows that the distribution of X relative to f is a convex combination of the distribution of X_1 relative to f_1 and X_2 relative to f_2. Let $Y = Y_1 \oplus Y_2 \in \mathscr{A}_1 \oplus \mathscr{A}_2$. It follows from (5.1) that if X_i does not interfere with Y_i relative to f_i, $i = 1, 2$, then X does not interfere with Y relative to f. Let $B = B_1 \dot{\cup} B_2 \in \Sigma_X$ and $A \in \Sigma_Y$. Then

$$\int_B |f_X[Y^{-1}(A)]|^2 \, d\mu_X = |a|^2 \int_{B_1} |f_{1X_1}[Y_1^{-1}(A_1)]|^2 \, d\mu_{X_1}$$
$$+ |b|^2 \int_{B_2} |f_{2X_2}[Y_2^{-1}(A_2)]|^2 \, d\mu_{X_2} \tag{5.2}$$

while

$$P_{X,f}(B) P_{X,f}[Y^{-1}(A)] = [|a|^2 P_{X_1, f_1}(B_1) + |b|^2 P_{X_2, f_2}(B_2)]$$
$$\times \{|a|^2 P_{X_1, f_1}[Y_1^{-1}(A_1)] + |b|^2 P_{X_2, f_2}[Y_2^{-1}(A_2)]\} \tag{5.3}$$

If X_i is independent of Y_i relative to f_i, $i=1,2$, then (5.2) gives

$$\int_B |f_X[Y^{-1}(A)]|^2\, d\mu_X = |a|^2 P_{X_1,f_1}(B_1) P_{X_1,f_1}[Y_1^{-1}(A_1)] + |b|^2 P_{X_2,f_2}(B_2) P_{X_2,f_2}[Y_2^{-1}(A_2)] \qquad (5.4)$$

In general, (5.3) and (5.4) do not coincide unless $ab=0$. Thus, even in the case of componentwise independence, X and Y are not independent in general.

The next result characterizes amplitude density direct sums. For $f \in \mathscr{D}(\mathscr{A}_1 \oplus \mathscr{A}_2)$ we write $\hat{f}_{X_i} = (f|\Omega_i)_{X_i}$, $i=1,2$.

Theorem 5.1. Let $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2$, $f \in \mathscr{D}(\mathscr{A})$ and assume that $\mathscr{D}(\mathscr{A}_1)$, $\mathscr{D}(\mathscr{A}_2) \neq \emptyset$. Then there exist $f_i \in \mathscr{D}(\mathscr{A}_i)$, $i=1,2$, $a, b \in \mathbb{C}$ with $|a|^2+|b|^2=1$ such that $f = af_1 \oplus bf_2$ if and only if (1) $\hat{f}_{X_1}=0$ a.e. $[\mu_{X_1}]$ for every $X_1 \in \mathscr{A}_1$ implies $f(\omega_1)=0$ for every $\omega_1 \in \Omega_1$, (2) $\hat{f}_{X_2}=0$ a.e. $[\mu_{X_2}]$ for every $X_2 \in \mathscr{A}_2$ implies $f(\omega_2)=0$ for every $\omega_2 \in \Omega_2$.

Proof. Suppose $f \in \mathscr{D}(\mathscr{A})$ and $f = af_1 \oplus bf_2$, $f_i \in \mathscr{D}(\mathscr{A}_i)$, $i=1,2$. Assume $\hat{f}_{X_1}=0$ a.e. $[\mu_{X_1}]$ for every $X_1 \in \mathscr{A}_1$. Since $af_1 = f|\Omega_1$, we have $af_{1X_1} = \hat{f}_{X_1}=0$ a.e. $[\mu_{X_1}]$ for every $X_1 \in \mathscr{A}_1$. Since $f_1 \in \mathscr{D}(\mathscr{A}_1)$, we have

$$|a| = |a|\ \|f_{1X_1}\| = \|af_{1X_1}\| = 0$$

so $a=0$. Hence, $|b|=1$ and $f = 0f_1 \oplus bf_2$. Then, for every $\omega_1 \in \Omega_1$, we have $f(\omega_1) = 0f_1(\omega_1)=0$. Therefore, (1) holds and in a similar way, (2) holds. Conversely, suppose $f \in \mathscr{D}(\mathscr{A})$ and (1) and (2) hold. Let $X = X_1 \oplus X_2 \in \mathscr{A}$. Then for $x_1 \in R(X_1)$ we have

$$f_X(x_1) = \int_{X^{-1}(x_1)} f\, d\mu_X^{x_1} = \int_{X_1^{-1}(x_1)} f\, d\mu_{X_1}^{x_1} = \hat{f}_{X_1}(x_1)$$

Similarly, $f_X(x_2) = \hat{f}_{X_2}(x_2)$ for every $x_2 \in R(X_2)$. Hence,

$$1 = \int_{R(X)} |f_X|^2\, d\mu_X = \int_{R(X_1)} |\hat{f}_{X_1}|^2\, d\mu_{X_1} + \int_{R(X_2)} |\hat{f}_{X_2}|^2\, d\mu_{X_2}$$

Define $a \geq 0$ by

$$a^2 = 1 - \int_{R(X_2)} |\hat{f}_{X_2}|^2\, d\mu_{X_2}$$

for a fixed $X_2 \in \mathscr{A}_2$. Then for every X_1, $Y_1 \in \mathscr{A}_1$ we have

$$\int_{R(X_1)} |\hat{f}_{X_1}|^2\, d\mu_{X_1} = \int_{R(Y_1)} |\hat{f}_{Y_1}|^2\, d\mu_{Y_1} = a^2$$

Similarly, there exists a $b \geq 0$ such that for every X_2, $Y_2 \in \mathscr{A}_2$ we have

$$\int_{R(X_2)} |\hat{f}_{X_2}|^2 \, d\mu_{X_2} = \int_{R(Y_2)} |\hat{f}_{Y_2}|^2 \, d\mu_{Y_2} = b^2$$

Then $a^2 + b^2 = 1$. If $a = 0$, then $\hat{f}_{X_1} = 0$ a.e. $[\mu_{X_1}]$ for every $X_1 \in \mathscr{A}_1$. Applying (1), we have $f(\omega_1) = 0$ for every $\omega_1 \in \Omega_1$. Let $f \in \mathscr{D}(\mathscr{A}_1)$ be arbitrary and define $f_2 : \Omega \to \mathbb{C}$ by $f_2(\omega_2) = f(\omega_2)$. To show that $f_2 \in \mathscr{D}(\mathscr{A}_2)$, let $X_2 \in \mathscr{A}_2$. Then for $x_2 \in R(X_2)$ we have

$$f_{2X_2}(x_2) = \int_{X_2^{-1}(x_2)} f_2 \, d\mu_{X_2}^{x_2} = \int_{X_2^{-1}(x_2)} f \, d\mu_{X_2}^{x_2} = \hat{f}_{X_2}(x_2)$$

Hence,

$$\int_{R(X_2)} |f_{2X_2}|^2 \, d\mu_{X_2} = \int_{R(X_2)} |\hat{f}_{X_2}|^2 \, \mathrm{d}\mu_{X_2} = b^2 = 1$$

Therefore, $f_2 \in \mathscr{D}(\mathscr{A}_2)$ and $f = 0 f_1 \oplus f_2$. Similarly, if $b = 0$, then $f = f_1 \oplus 0 f_2$ for some $f_i \in \mathscr{D}(\mathscr{A}_i)$, $i = 1, 2$. Now suppose a, $b > 0$. Define $f_1 : \Omega_1 \to \mathbb{C}$ by $f_1(\omega_1) = f(\omega_1)/a$ and f_2: $\Omega_2 \to \mathbb{C}$ by $f_2(\omega_2) = f(\omega_2)/b$. To show that $f_1 \in \mathscr{D}(\mathscr{A}_1)$, let $X_1 \in \mathscr{A}_1$. Then for $x_1 \in R(X_1)$ we have

$$f_{1X_1} = \int_{X_1^{-1}(x_1)} f_1 \, d\mu_{X_1}^{x_1} = \frac{1}{a} \int_{X_1^{-1}(x_1)} f \, d\mu_{X_1}^{x_1} = \frac{1}{a} \hat{f}_{X_1}(x_1)$$

Hence,

$$\int_{R(X_1)} |f_{1X_1}|^2 \, d\mu_{X_1} = \frac{1}{a^2} \int_{R(X_1)} |\hat{f}_{X_1}|^2 \, d\mu_{X_1} = 1$$

Similarly, $f_2 \in \mathscr{D}(\mathscr{A}_2)$. Moreover, $f = af_1 \oplus bf_2$. ■

Corollary 5.2. Let $f \in \mathscr{D}(\mathscr{A}_1 \oplus \mathscr{A}_2)$ and suppose there exist $X_i \in \mathscr{A}_i$ such that $\hat{f}_{X_i} \neq 0$ a.e. $[\mu_{X_i}]$, $i = 1, 2$. Then there exist $f_i \in \mathscr{D}(\mathscr{A}_i)$, $i = 1, 2$, a, $b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$ such that $f = af_1 \oplus bf_2$.

Corollary 5.3. Let $f \in \mathscr{H}(\mathscr{A}_1 \oplus \mathscr{A}_2)$ and suppose there exist $X_i \in \mathscr{A}_i$ such that $\hat{f}_{X_i} \neq 0$ a.e. $[\mu_{X_i}]$, $i = 1, 2$. Then there exist $f_i \in \mathscr{H}(\mathscr{A}_i)$, $i = 1, 2$, such that $f = f_1 \oplus f_2$.

The next result shows that the decomposition $f = af_1 \oplus bf_2$ is essentially unique.

Lemma 5.4. Let $f_i, f_i' \in \mathscr{D}(\mathscr{A}_i)$, $i = 1, 2$, and suppose

$$af_1 \oplus bf_2 = a'f_1' \oplus b'f_2'$$

where a, $b \neq 0$. Then there exist c, $d \in \mathbb{C}$ with $|c| = |d| = 1$ such that $f_1 = cf_1'$, $f_2 = df_2'$ and $a' = ac$, $b' = bd$.

Proof. Since $af_1(\omega_1)=a'f_1'(\omega_1)$ for all $\omega_1\in\Omega_1$ we have $f_1=a'f_1'/a$. Hence,

$$1=\|f_1\|=\left|\frac{a'}{a}\right|\|f_1'\|=\left|\frac{a'}{a}\right|$$

Letting $c=a'/a$, we have $f_1=cf_1'$ and $a'=ac$. A similar result holds for f_2. ■

We now give an example of an $f\in\mathscr{D}(\mathscr{A}_1\oplus\mathscr{A}_2)$ which is not of the form $f=af_1\oplus bf_2$, $f_i\in\mathscr{D}(\mathscr{A}_i)$, $i=1,2$.

Example 3. Let $\Omega_1=\{\omega_1,\omega_2\}$, $X_1(\omega_1)=X_1(\omega_2)=x_1$, with counting measure on the fiber and range. Let $\Omega_2=\{\omega_3\}$, $X_2(\omega_3)=x_2$, again with counting measure on the fiber and range. Form the catalogs $\mathscr{A}_1=\{X_1\}$, $\mathscr{A}_2=\{X_2\}$. Define $f:\Omega_1\dot\cup\Omega_2$ by $f(\omega_1)=-f(\omega_2)=f(\omega_3)=1$. Then $f\in\mathscr{D}(\mathscr{A}_1\oplus\mathscr{A}_2)$. Indeed, $\hat f_{X_1}=0$ and $\hat f_{X_2}=1$. Hence, for $X=X_1\oplus X_2$ we have

$$\sum_{x\in R(X)}|f_X(x)|^2=1$$

Now $\hat f_{X_1}=0$ for all $X_1\in\mathscr{A}_1$, yet $f(\omega)\neq 0$ for all $\omega\in\Omega_1$. By Theorem 5.1, $f\neq af_1\oplus bf_2$ for $f_i\in\mathscr{D}(\mathscr{A}_i)$, $i=1,2$. ■

We now consider sectors in the direct sum $\mathscr{A}_1\oplus\mathscr{A}_2$. Since null amplitudes are identified with the zero amplitude, it follows from the proof of Theorem 5.1 that modulo a null amplitude, $f\in\mathscr{H}(\mathscr{A}_1\oplus\mathscr{A}_2)$ if and only if $f=f_1\oplus f_2$, where $f_i\in\mathscr{H}(\mathscr{A}_i)$, $i=1,2$. Moreover, $\hat f_{X_i}=f_{iX_i}$ for every $X_i\in\mathscr{A}_i$, $i=1,2$. Clearly, f_1 and f_2 are unique.

Lemma 5.5. If f, $g\in\mathscr{H}(\mathscr{A}_1\oplus\mathscr{A}_2)$, then $f s g$ if and only if $f_1 s g_1$ and $f_2 s g_2$.

Proof. Suppose $f_1 s g_1, f_2 s g_2$. For $X_1\in\mathscr{A}_1$, $X_2\in\mathscr{A}_2$, and $X=X_1\oplus X_2$ we have

$$\int_{R(X)}f_X\bar g_X\,d\mu_X=\int_{R(X_i)}f_{1X_1}\bar g_{1X_1}\,d\mu_{X_1}+\int_{R(X_2)}f_{2X_2}\bar g_{2X_2}\,d\mu_{X_2}$$

and the right side is independent of $X_1\in\mathscr{A}_1$, $X_2\in\mathscr{A}_2$. Hence, $f s g$. Conversely, suppose $f s g$ and X_1, $Y_1\in\mathscr{A}_1$. Let $X_2\in\mathscr{A}_2$ and define $X=X_1\oplus X_2$ and $Y=Y_1\oplus X_2$ in $\mathscr{A}_1\oplus\mathscr{A}_2$. Then

$$\begin{aligned}\int_{R(X_1)}f_{1X_1}\bar g_{1X_1}\,d\mu_{X_1}&=\int_{R(X)}f_X\bar g_X\,d\mu_X-\int_{R(X_2)}f_{2X_2}\bar g_{2X_2}\,d\mu_{X_2}\\&=\int_{R(Y)}f_Y\bar g_Y\,d\mu_Y-\int_{R(X_2)}f_{2X_2}\bar g_{2X_2}\,d\mu_{X_2}\\&=\int_{R(Y_1)}f_{1Y_1}\bar g_{1Y_1}\,d\mu_{Y_1}\end{aligned}$$

Hence $f_1 \, s \, g_1$ and similarly $f_2 \, s \, g_2$. ■

For $A_i \subseteq \mathcal{H}(\mathcal{A}_i)$, $i = 1, 2$, we use the notation

$$A_1 \oplus A_2 = \{f_1 \oplus f_2 : f_1 \in A_1, f_1 \in A_2\}$$

Theorem 5.6. $M \subseteq \mathcal{H}(\mathcal{A}_1 \oplus \mathcal{A}_2)$ is a sector if and only if $M = M_1 \oplus M_2$ where $M_i \in \mathcal{M}(\mathcal{A}_i)$, $i = 1, 2$.

Proof. Suppose $M \in \mathcal{M}(\mathcal{A}_1 \oplus \mathcal{A}_2)$ and let

$$M_i = \{f_i : f \in M\} \subseteq \mathcal{H}(\mathcal{A}_i), \qquad i = 1, 2$$

Then $M \subseteq M_1 \oplus M_2$ and by Lemma 5.5, $M_i \subseteq M_i^s$, $i = 1, 2$. Suppose $g_1 \in M_1^s$. If $g_2 \in M_2$, let $g = g_1 \oplus g_2$. Then by Lemma 5.5, $g \in M^s = M$. Hence, $g_1 \in M_1$. Therefore, $M_1 = M_1^s$, so $M_1 \in \mathcal{M}(\mathcal{A}_1)$ and similarly $M_2 \in \mathcal{M}(\mathcal{A}_2)$. Finally, if $g_i \in M_i$, $i = 1, 2$, then

$$g = g_1 \oplus g_2 \in M^s = M$$

Hence, $M = M_1 \oplus M_2$. Conversely, let $M_i \in \mathcal{M}(\mathcal{A}_i)$, $i = 1, 2$, and let $M = M_1 \oplus M_2$. By Lemma 5.5, $M \subseteq M^s$. Let $g \in M^s$, with $g = g_1 \oplus g_2$. Again by Lemma 5.5, $g_i \in M_i^s = M_i$, $i = 1, 2$. Hence, $g \in M$. Therefore, $M = M^s$, so $M \in \mathcal{M}(\mathcal{A}_1 \oplus \mathcal{A}_2)$. ■

For inner product spaces H_1, H_2, let $H_1 \hat{\oplus} H_2$ be the usual inner product space direct sum. That is,

$$H_1 \hat{\oplus} H_2 = \{(\psi_1, \psi_2) : \psi_1 \in H_1, \psi_2 \in H_2\}$$

where addition and scalar multiplication are defined componentwise and

$$\langle (\psi_1, \psi_2), (\phi_1, \phi_2) \rangle = \langle \psi_1, \phi_1 \rangle + \langle \psi_2, \phi_2 \rangle$$

Lemma 5.7. If $M_i \in \mathcal{M}(\mathcal{A}_i)$, $i = 1, 2$, then the map $J: M_1 \oplus M_2 \to M_1 \hat{\oplus} M_2$ given by $J(f_1 \oplus f_2) = (f_1, f_2)$ is an isomorphism.

Proof. Clearly, J is a linear bijection. Moreover, for $X = X_1 \oplus X_1 \in \mathcal{A}_1 \oplus \mathcal{A}_2$ and $f_i, g_i \in M_i$, $i = 1, 2$, we have

$$\begin{aligned} \langle f_1 \oplus f_1, g_1 \oplus g_2 \rangle &= \int_{R(X)} (f_1 \oplus f_2)_X \overline{(g_1 \oplus g_2)_X} \, d\mu_X \\ &= \int_{R(X_1)} f_{1X_1} \bar{g}_{1X_1} \, d\mu_{X_1} + \int_{R(X_2)} f_{2X_2} \bar{g}_{2X_2} \, d\mu_{X_2} \\ &= \langle (f_1, f_2), (g_1, g_2) \rangle = \langle J(f_1 \oplus f_2), J(g_1 \oplus g_2) \rangle \end{aligned}$$ ■

6. CARTESIAN PRODUCTS

If A and B are sets, we denote their Cartesian product $A \times B$ by AB. We denote an element $(a, b) \in AB$ by ab. For measurements $X \in \hat{\mathscr{A}}(\Omega)$ and $Y \in \hat{\mathscr{A}}(\Omega')$ we define the *Cartesian product* XY of X and Y as the map

$$XY: \quad \Omega\Omega' \to R(X)R(Y)$$

given by $XY(\omega\omega') = X(\omega)Y(\omega')$. For $xy \in R(XY) = R(X)R(Y)$, the fiber $(XY)^{-1}(xy) = X^{-1}(x)Y^{-1}(y)$ is the base space of the measure space

$$(X^{-1}(x)Y^{-1}(y), \Sigma_X^x \times \Sigma_Y^y, \mu_X^x \times \mu_Y^y)$$

Moreover, the range $R(XY)$ is the base space of the measure space

$$(R(X)R(Y), \Sigma_X \times \Sigma_Y, \mu_X \times \mu_Y)$$

Equipped with these measure structures, XY becomes a measurement on $\Omega\Omega'$. It is straightforward to extend this definition to form the Cartesian product of any finite number of measurements.

Let $\mathscr{A} \subseteq \hat{\mathscr{A}}(\Omega)$, $\mathscr{B} \subseteq \hat{\mathscr{A}}(\Omega')$ be catalogs. We then define the catalog $\mathscr{A}\mathscr{B} \subseteq \hat{\mathscr{A}}(\Omega\Omega')$ by

$$\mathscr{A}\mathscr{B} = \{XY: X \in \mathscr{A}, Y \in \mathscr{B}\}$$

For $f \in \mathscr{H}(\mathscr{A})$, $g \in \mathscr{H}(\mathscr{B})$, we define $fg: \Omega\Omega' \to \mathbb{C}$ by $fg(\omega\omega') = f(\omega)g(\omega')$, where the right side is the usual complex product. If $XY \in \mathscr{A}\mathscr{B}$, we have

$$\begin{aligned}(fg)_{XY}(xy) &= \int_{(XY)^{-1}(xy)} fg\, d\mu_{XY}^{xy} = \int_{X^{-1}(x)} f\, d\mu_X^x \int_{Y^{-1}(y)} g\, d\mu_Y^y \\ &= f_X(x)g_Y(y)\end{aligned}$$

Hence,

$$\int_{R(XY)} |(fg)_{XY}(xy)|^2\, d\mu_{XY} = \int_{R(X)} |f_X|^2\, d\mu_X \int_{R(Y)} |f_Y|^2\, d\mu_Y$$

It follows that $fg \in \mathscr{H}(\mathscr{A}\mathscr{B})$ and $\|fg\| = \|f\|\,\|g\|$. In particular, if $f \in \mathscr{D}(\mathscr{A})$, $g \in \mathscr{D}(\mathscr{B})$, then $fg \in \mathscr{H}(\mathscr{A}\mathscr{B})$.

If $C \in \mathscr{E}(XY, fg)$, then

$$(fg)_{XY}(C)(xy) = \int_{C \cap (XY)^{-1}(xy)} fg\, d\mu_{XY}^{xy}$$

In general, this expression cannot be simplified unless C is a product event $C = AB$, $A \in \mathscr{E}(X, f)$. $B \in \mathscr{E}(Y, g)$. In this case

$$\begin{aligned}(fg)_{XY}(AB)(xy) &= \int_{A \cap X^{-1}(x)} f\, d\mu_X^x \int_{B \cap Y^{-1}(y)} g\, d\mu_Y^y \\ &= f_X(A)(x)f_Y(B)(y)\end{aligned} \tag{6.1}$$

Applying (6.1) then gives

$$P_{XY,fg}(AB) = P_{X,f}(A)P_{Y,g}(B) \tag{6.2}$$

Suppose X, $X' \in \mathscr{A}$, Y, $Y' \in \mathscr{B}$, X does not interfere with X' relative to f, Y does not interfere with Y' relative to g, and $AB \in \mathscr{E}(X'Y')$. Then from (6.2) we have

$$\begin{aligned} P_{XY,fg}[(X'Y')^{-1}(AB)] &= P_{XY,fg}[(X')^{-1}(A)(Y')^{-1}(B)] \\ &= P_{X',f}(A)P_{Y',g}(B) \\ &= P_{X'Y',fg}(AB) \end{aligned}$$

However, every $C \in \mathscr{E}(X'Y')$ need not be a product event $C = AB$ and in general we may have

$$P_{XY,fg}[(X'Y')(C)] \neq P_{X'Y',fg}(C)$$

so XY can interfere with $X'Y'$ relative to fg. A similar observation holds for independence.

If $f \in \mathscr{H}(\mathscr{A}_1\mathscr{A}_2)$ has the form $f = f_1 f_2$, $f_i \in \mathscr{H}(\mathscr{A}_i)$, $i = 1, 2$, we call f a *product amplitude*. For f, $g \in \mathscr{H}(\mathscr{A})$, we write $f \perp g$ if $f s g$ and $\langle f, g\rangle = 0$.

Theorem 6.1. Let f, $g \in \mathscr{H}(\mathscr{A}_1\mathscr{A}_2)$ be product amplitudes. Then $f s g$ if and only if one of the following conditions holds: (a) $f_1 s g_1$ and $f_2 s g_2$, (b) $f_1 \perp g_1$, (c) $f_2 \perp g_2$.

Proof. It is clear that $f s g$ if and only if

$$\langle f_{1X_1}, g_{1X_1}\rangle\langle f_{2X_2}, g_{2X_2}\rangle = \langle f_{1Y_1}, g_{1Y_1}\rangle\langle f_{2Y_2}, g_{2Y_2}\rangle \tag{6.3}$$

for every X_1, $Y_1 \in \mathscr{A}_1$, X_2, $Y_2 \in \mathscr{A}_2$. If (a), (b), or (c) holds, then (6.3) holds, so $f s g$. Conversely, suppose $f s g$. If $f_1 \not\perp g_1$ and $f_2 \not\perp g_2$, then there exists an $X_1 \in \mathscr{A}_1$ such that $\langle f_{1X_1}, g_{1X_1}\rangle \neq 0$. Letting $Y_1 = X_1$ and applying (6.3) gives $f_2 s g_2$. Similarly, $f_1 s g_1$. Now suppose $f_1 \not s g_1$. Then there exist X_1, $Y_1 \in \mathscr{A}_1$ such that

$$\langle f_{1X_1}, g_{1X_1}\rangle \neq \langle f_{1Y_1}, g_{1Y_1}\rangle$$

Letting $X_2 = Y_2$ and applying (6.3) gives $\langle f_{2X_2}, g_{2X_2}\rangle = 0$. Since $X_2 \in \mathscr{A}_2$ is arbitrary, $f_2 \perp g_2$. Similarly, $f_2 \not s g_2$ implies $f_1 \perp g_1$. ∎

For $M_i \in \mathscr{M}(\mathscr{A}_i)$, $i = 1, 2$, we define

$$M_1M_2 = \{f_1 f_2 : f_i \in M_i, i = 1, 2\}$$

It follows from Theorem 6.1 that M_1M_2 is a subset of a sector of $\mathscr{H}(\mathscr{A}_1\mathscr{A}_2)$. In general, M_1M_2 is not itself a sector. Denoting the tensor product of M_1 and M_2 by $M_1 \otimes M_2$, it is clear that the map $K: M_1M_2 \to M_1 \otimes M_2$ defined by $K(f_1 f_2) = f_1 \otimes f_2$ extends to a unique isomorphism from span M_1M_2 into $M_1 \otimes M_2$.

Our next result characterizes product amplitudes. For $f \in \mathcal{H}(\mathcal{A}_1\mathcal{A}_2)$, $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$, define $f_{\omega_2}(\omega_1) = f(\omega_1, \omega_2)$ and $f_{\omega_1}(\omega_2) = f(\omega_1, \omega_2)$.

Theorem 6.2. Let $f \in \mathcal{H}(\mathcal{A}_1\mathcal{A}_2)$. Then f is a product amplitude if and only if $f_{\omega_2} \in \mathcal{H}(\mathcal{A}_1)$, $f_{\omega_1} \in \mathcal{H}(\mathcal{A}_2)$ for every $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ and for every ω_1, $\omega_1' \in \Omega_1$, ω_2, $\omega_2' \in \Omega_2$ we have

$$f(\omega_1\omega_2)f(\omega_1'\omega_2') = f(\omega_1\omega_2')f(\omega_1'\omega_2)$$

Proof. Suppose f is a product amplitude and $f = f_1 f_2$. Then $f_{\omega_2} = f_2(\omega_2)f_1$ and $f_{\omega_1} = f_1(\omega_1)f_2$ are amplitudes and

$$\begin{aligned} f(\omega_1\omega_2)f(\omega_1'\omega_2') &= f_1(\omega_1)f_2(\omega_2)f_1(\omega_1')f_2(\omega_2') \\ &= f(\omega_1\omega_2')f(\omega_1'\omega_2) \end{aligned}$$

Conversely, suppose f satisfies the conditions of the theorem. If $f = 0$, then clearly f is a product amplitude. Otherwise, there exist $\omega_1' \in \Omega_1$, $\omega_2' \in \Omega_2$ such that $f(\omega_1'\omega_2') \neq 0$. Then for every $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ we have

$$f(\omega_1\omega_2) = \frac{f(\omega_1\omega_2')f(\omega_1'\omega_2)}{f(\omega_1'\omega_2')} = \frac{1}{f(\omega_1'\omega_2')} f_{\omega_2'}(\omega_1) f_{\omega_1'}(\omega_2) \quad \blacksquare$$

Although we shall not pursue this matter here, it is interesting to observe that quantum field theory can be formulated within this framework. If $\mathcal{A} \subseteq \hat{\mathcal{A}}(\Omega)$ is a catalog, we write $\mathcal{A}^n = \mathcal{A}\mathcal{A} \cdots \mathcal{A}$, where there are n factors on the right side. Let $\Omega_0 = \{\omega_0\}$ be a singleton set and define the measurement $X_0 : \Omega \to \{x_0\}$ by $X(\omega_0) = x_0$ with counting measure on the fiber and range. We define $\mathcal{A}^0 = \{X_0\}$. The *Fock catalog over* $\mathcal{A}$ is defined by

$$\Gamma(\mathcal{A}) = \mathcal{A}^0 \oplus \mathcal{A}^1 \oplus \mathcal{A}^2 \oplus \cdots$$

If $f \in \mathcal{H}(\mathcal{A})$, we define the *exponential* (or *coherent*) *amplitude* $\exp(f)$ by

$$\exp(f) = 1 \oplus f \oplus \frac{f^2}{2!} \oplus \cdots \oplus \frac{f^n}{n!} \oplus \cdots$$

where $f^n = ff \cdots f$ (n factors). It is easy to see that $f s g$ if and only if $\exp(f)\, s\, \exp(g)$. It follows that if $M \in \mathcal{M}(\mathcal{A})$, then

$$\exp(M) = \{\exp(f) : f \in M\}$$

is contained in a sector of $\Gamma(\mathcal{A})$. We leave the further development of this theory to future work.

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